TNEB Syllabus-Class 11-Physics-UNIT 5

MOTION OF SYSTEM OF PARTICLES AND RIGID BODIES

ROTATIONAL DYNAMICS

This involves **obtaining the relation between the intrinsic properties of the body such as center of mass, moment of inertia etc**.
All the parameters that are involved in study of a body in motion in one or two dimensions, the same applies for a body undergoing rotational motion.

1. Center of Mass

the overall motion of a system can be described in terms of a *special point* called the **center of mass of the system**.

The translational motion of the ***center of mass of the system*** *is the same as if all the mass of the system were concentrated at that point.*

Eg. When a bulk object (say a bat) is thrown at an angle in air as shown in Figure 1 below ;

All the points of the body do not take a parabolic path as observed in the motion of a particle.

Actually, **only one point** *takes the parabolic path* and *all the other points take different paths.*

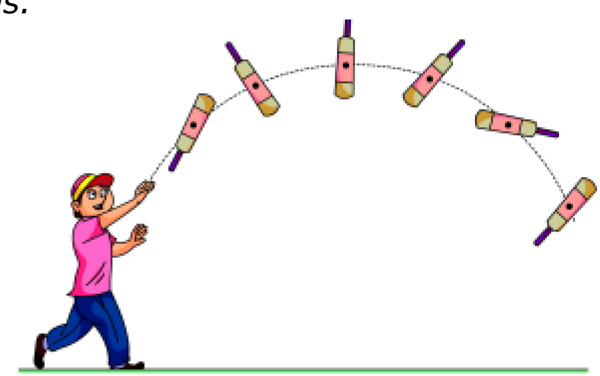


Fig. 1- Centre of mass tracing the path of a parabola

The one point that takes the parabolic path is the very special point called Centre of Mass (CM) of the body.
Its motion is like the motion of a single point that is thrown.
The center of mass of a body is defined as a point where the entire mass of the body appears to be concentrated.
Therefore, this point (CM)can represent the entire body.

http://physics.bu.edu/~duffy/HTML5/centerofmass.html

http://physics.bu.edu/~duffy/classroom.html

https://ophysics.com/r1a.html

https://www.youtube.com/watch?v=1lhAD88fWG8

For bodies of regular shape and uniform mass distribution, the centre of mass is at the geometric centre of the body.
As examples,

- ✓ for a **circle and sphere**, the centre of mass is at their centres;
- ✓ for **square and rectangle**, at the point their diagonals meet;
- ✓ for **cube and cuboid**, it is at the point where their body diagonals meet.
- ✓ For other bodies, the centre of mass has to be determined using some methods.
- ✓ The centre of mass could be well within the body and in some cases outside the body as well

Centre of Mass for Distributed Point Masses

A point mass is a hypothetical point particle which has nonzero mass and no size or shape.
To find the centre of mass for a collection of n point masses, say, m_1, m_2 , m_3 . . . m_n we have to first choose an origin and an appropriate coordinate system as shown in **Figure 2**.

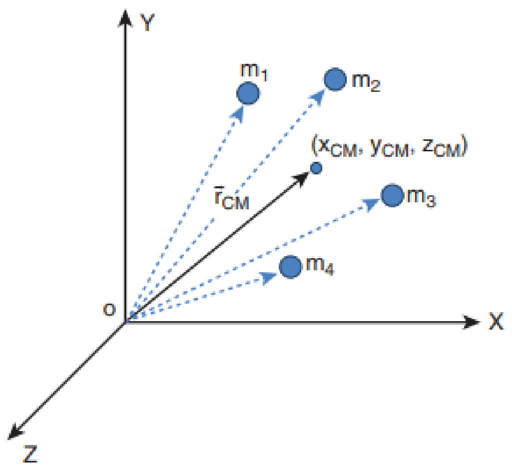


Figure 2 Centre of mass for distributed point masses

Let, x_1, x_2 , x_3 . . . x_n be the X-coordinates of the positions of these point masses in the X direction from the origin.

The equation for the x coordinate of the centre of mass is

$$x_{CM} = \frac{\sum m_1 x_1}{\sum m_1}$$

where $\sum m_1$is the total mass M of all the particles, $(\sum m_1 = M)$.

$$x_{CM} = \frac{\sum m_1 x_1}{M} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5.1)$$

Similarly, we can also find y and z coordinates of the centre of mass for these distributed point masses as indicated in Figure 2

$$y_{CM} = \frac{\sum m_1 y_1}{M} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5.2)$$

$$z_{CM} = \frac{\sum m_1 z_1}{M} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5.3)$$

Hence, the position of centre of mass of these point masses in a Cartesian coordinate system is (x_{CM}, y_{CM}, z_{CM}).
In general, the position of centre of mass can be written in a vector form as

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5.4)$$

where, $\boldsymbol{\vec{r}_{CM} = x_{CM}\hat{\imath} + y_{CM}\hat{\jmath} + z_{CM}\hat{k}}$

is the **position vector** of the **center of mass**

and

$$\boldsymbol{\vec{r}_i = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}}$$

is the **position vector** of the **distributed point mass;**

where, $\hat{\imath}, \hat{\jmath}\ and\ \hat{k}$ are the unit vectors along X, Y and Z-axes respectively

Centre of Mass of Two Point Masses

(i) *When the masses are on positive X-axis*

The origin is taken arbitrarily so that the masses m_1 and m_2 are at positions x_1 and x_2 on the positive X-axis as shown in Figure 3(a).

The **center of mass** will also be on the positive X-axis **at x_{CM}** as given by the equation,

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

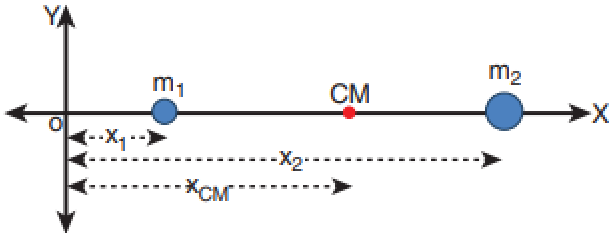


Figure 3(a) When the masses are on positive X axis

(ii) *When the origin coincides with any one of the masses:*

The calculation gets minimized, if the origin of the coordinate system is made to coincide with any one of the masses as shown in Figure 3(b). When the origin coincides with the point mass m_1, its position x_1 is zero, (i.e. $x_1 = 0$). Then,

$$x_{CM} = \frac{m_1(0) + m_2 x_2}{m_1 + m_2}$$

The equation further simplifies as,

$$x_{CM} = \frac{m_2 x_2}{m_1 + m_2}$$

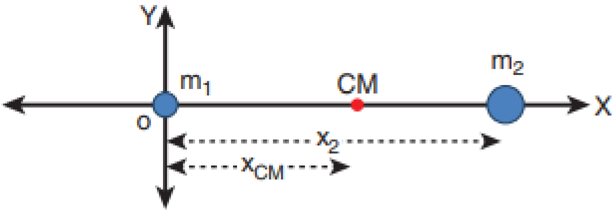


Figure 3(b) When the origin coincides with any one of the masses

(iii) When the origin coincides with the center of mass itself:

If the origin of the coordinate system is made to coincide with the center of mass,
then, x_{CM}=0 and the mass m1 is found to be on the negative X-axis as shown in Figure 3(c). Hence, its position x_1 is negative, (i.e.$-x_1$).

$$0 = \frac{m_1(-x_1) + m_2x_2}{m_1 + m_2}$$

$$0 = m_1(-x_1) + m_2x_2$$

$$\boldsymbol{m_1x_1 = m_2x_2}$$

The equation given above is known as **principle of moments**.

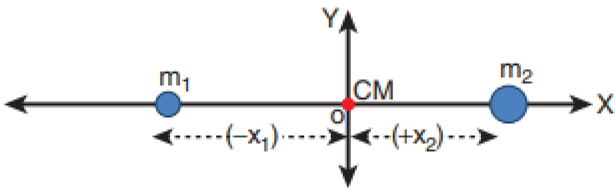
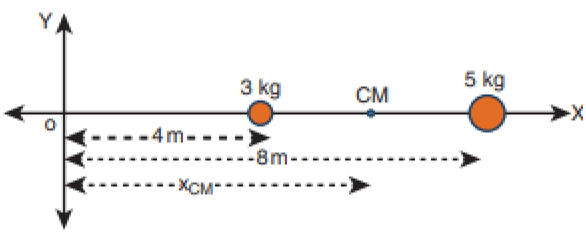


Figure 3(c) When the origin coincides with the center of mass itself

EXAMPLE 5.1
Two point masses 3 kg and 5 kg are at 4 m and 8 m from the origin on X-axis. Locate the position of center of mass of the two point masses
(i) from the origin and
(ii) (ii) from 3 kg mass

Solution
Let us take, m_1 = 3 kg and m_2 = 5 kg
(i) *To find center of mass from the origin*:
The point masses are at positions, x_1 = 4 m, x_2 = 8 m from the origin along X axis.



The centre of mass x_{CM} can be obtained using the general equation 5.4 as given below

$$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{M}$$

In this case, as *the masses are on positive X-axis*

$\vec{r}_{CM}$ *is represented by* x_{CM}

m_i *is represented by* m_1 *and* m_2 and

$M = m_1 + m_2$

$$\therefore x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

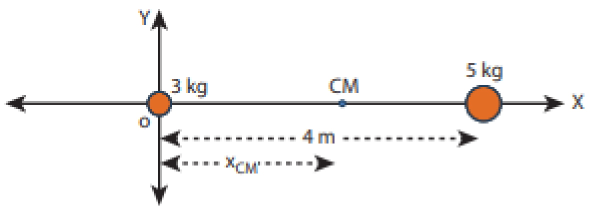
$$x_{CM} = \frac{(3 \times 4) + (5 \times 8)}{3 + 5}$$

$$x_{CM} = \frac{12 + 40}{8} = \frac{52}{8} = 6.5m$$

The center of mass is located 6.5 m from the origin on X-axis

(ii) *To find the center of mass from 3 kg mass*:

The origin is shifted to 3 kg mass along X-axis. The position of 3 kg point mass is zero (x_1 = 0) and the position of 5 kg point mass is 4 m from the shifted origin (x_2 = 4 m).



$$x_{CM} = \frac{(3 \times 0) + (5 \times 4)}{8} = \frac{20}{8} = 2.5m$$

The centre of mass is located 2.5m from 3 kg point mass, (and 1.5 m from the 5 kg point mass) on X-axis.

- This result shows that the centre of mass is located closer to larger mass.

- If the origin is shifted to the centre of mass, then the principle of moments holds good.

$$m_1 x_1 = m_2 x_2 \ \ i.e. 3 \times 2.5 = 5 \times 1.5 \, i.e \ 7.5 = 7.5$$

EXAMPLE 5.2

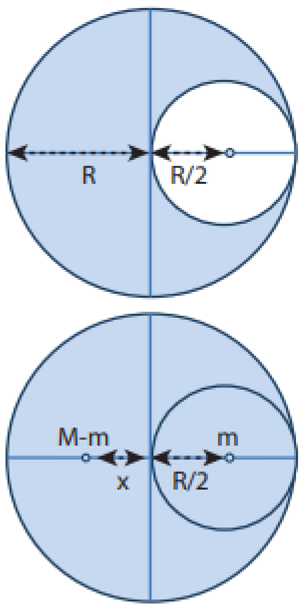
From a uniform disc of radius R, a small disc of radius $\frac{R}{2}$ is cut and removed as shown in the diagram.

Find the center of mass of the remaining portion of the disc.

Solution

Let us consider the mass of the uncut full disc be M.
Its centre of mass would be at the geometric center of the disc on which the origin coincides.

Let the mass of the small disc cut and removed be **m** and its centre of mass is at a position $\frac{R}{2}$ to the right of the origin as shown in the figure.



Hence, the remaining portion of the disc should have its center of mass to the left of the origin; say, at a distance x.
We can write from the principle of moments,

$$(M - m)x = (m)\frac{R}{2}$$

$$x = \left[\frac{m}{(M-m)}\right]\frac{R}{2}$$

If σ is the surface mass density (i.e. mass per unit surface area),

$$\sigma = \frac{M}{\pi R^2}$$

then, the mass m of small disc is

$$m = surface\ mass\ density \times surface\ area$$

$$m = \sigma \times \pi\left(\frac{R}{2}\right)^2$$

$$m = \left(\frac{M}{\pi R^2}\right) \times \pi\left(\frac{R}{2}\right)^2 = \frac{M}{\cancel{\pi}\cancel{R^2}} \times \cancel{\pi}\frac{\cancel{R^2}}{4} = \frac{M}{4}$$

substituting m in the expression for x

$$x = \left[\frac{m}{(M-m)}\right]\frac{R}{2} = \left[\frac{\frac{M}{4}}{(M-\frac{M}{4})}\right]\frac{R}{2} = \frac{\frac{M}{4}}{\left(\frac{3M}{4}\right)} \times \left(\frac{R}{2}\right)$$

i.e $x = \frac{R}{6}$

The center of mass of the remaining portion is at a distance R/6 to the left from the center of the disc.

Motion of Centre of Mass

When a rigid body moves, its centre of mass will also move along with the body. For kinematic quantities like velocity v_{CM} and acceleration a_{CM} of the centre of mass, we can differentiate the expression for position of centre of mass with respect to time once and twice respectively.

For simplicity, let us take the motion along X direction only.

$$\vec{v}_{CM} = \frac{d\vec{x}_{CM}}{dt} = \frac{\sum m_i \left(\frac{d\vec{x}_i}{dt}\right)}{\sum m_i}$$

$$\vec{v}_{CM} = \frac{\sum m_i \vec{v}_i}{\sum m_i} \quad (5.7)$$

$$\vec{a}_{CM} = \frac{d}{dt}\left(\frac{d\vec{x}_{CM}}{dt}\right) = \left(\frac{d\vec{v}_{CM}}{dt}\right) = \frac{\sum m_i \left(\frac{d\vec{v}_i}{dt}\right)}{\sum m_i}$$

$$\vec{a}_{CM} = \frac{\sum m_i \vec{a}_i}{\sum m_i} \quad (5.8)$$

In the absence of external force, i.e. Fext = 0, the individual rigid bodies of a system can move or shift only due to the internal forces.
This will not affect the position of the centre of mass.
This means that the centre of mass will be in a state of rest or uniform motion.
Hence $\vec{v}_{CM}$ will be zero when centre of mass is at rest and constant when centre of mass has uniform motion
$\vec{v}_{CM} = 0 \ or \ \vec{v}_{CM}$=constant
There will be no acceleration of centre of mass, $\vec{a}_{CM} = 0$

From equation (5.7) and 5.8,

$$\vec{v}_{CM} = \frac{\sum m_i \vec{v}_i}{\sum m_i} = 0 \text{ (or) } \vec{v}_{CM} = \text{constant}$$

It implies

$$\vec{a}_{CM} = \frac{\sum m_i \vec{a}_i}{\sum m_i} = 0$$

Here, the individual particles may still move with their respective velocities and accelerations due to internal forces.

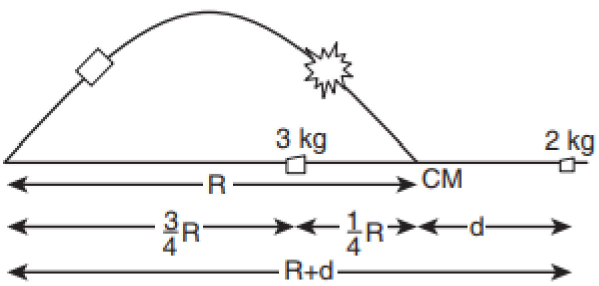
In the presence of external force, (i.e. Fext ≠ 0), the center of mass of the system will accelerate as given by the following equation.

$$\vec{F}_{ext} = \left(\sum m_i\right)\vec{a}_{CM}; \quad \vec{F}_{ext} = M\vec{a}_{CM}; \quad \vec{a}_{CM} = \frac{\vec{F}_{ext}}{M}$$

Centre of mass in explosions:

Many a times rigid bodies are broken into fragments.
If an explosion is caused by the internal forces in a body which is at rest or in motion, the state of the centre of mass is not affected.
It continues to be in the same state of rest or motion.
But, the kinematic quantities of the fragments get affected. If the explosion is caused by an external agency, then the kinematic quantities of the centre of mass as well as the fragments get affected.
Example
A projectile of mass 5 kg, in its course of motion explodes on its own into two fragments. One fragment of mass 3 kg falls at three fourth of the range R of the projectile. Where will the other fragment fall?

Solution
It is an explosion of its own without any external influence. After the explosion, the centre of mass of the projectile will continue to complete the parabolic path even though the fragments are not following the same parabolic path.
After the fragments have fallen on the ground, the centre of mass rests at a distance R (the range) from the point of projection as shown in the diagram



If the origin is fixed to the final position of the center of mass, the principle of moments holds good.

$$m_1 x_1 = m_2 x_2$$

where, m1 = 3 kg, m2 = 2 kg, x1 = 1/4 R. The value of x2 = d

$$3 \times \frac{1}{4} R = 2 \times d;$$

$$d = \frac{3}{8} R$$

The distance between the point of launching and the position of 2 kg mass is R+d

$$R + d = R + \frac{3}{8} R = \frac{11}{8} R = 1.375R$$

The other fragment falls at a distance of 1.375R from the point of launching. (Here R is the range of the projectile.)

There is one major difference between **mass and moment of inertia**. ***Mass*** *is an inherent property of an object***.**

The moment of inertia of an object depends on your choice of rotation axis.

Therefore, there is no single value of the moment of inertia for an object. There is a ***minimum value of the moment of inertia***, which is that calculated about an axis passing through the **center of mass of the object**